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CONCERNING DU BOIS-REYMOND'S TWO RELATIVE INTEGRABILITY THEOREMS.

BY ELIAKIM HASTINGS MOORE.

1. Introduction. The two theorems of du Bois-Reymond to be considered are suggestively (and incompletely) expressed in the form:

I. *A continuous function of (properly) integrable functions is integrable.*

II. *An integrable function of an integrable function is integrable.*

Du Bois-Reymond announced the first theorem in 1880 (*Math. Annalen*, vol. 16, p. 112), and published a proof two years later (*ibid.*, vol. 20, pp. 122–124), and in the latter connection he announced the second theorem.

I have found no further reference to the second theorem; in §5 I shall show by a simple example that it is not true.

When one thinks, in the first theorem, of the integral of the compound function as the limit of a sum, I shall show in §§2, 3, 4, that the range of values of the terms of the sum may be extended without alteration of the integral-limit.

In the case of the product of two integrable functions (for which case the original theorem was proved by du Bois-Reymond in 1875, in vol. 75 of the *Journal für Math.*, p. 24), this extension was made by Dini (1878: Dini-Lüroth, *Functionen einer reellen Veränderlichen*, §190, p. 347); his process of proof is not applicable to the general case. Du Bois-Reymond's general proof (1882) is however capable of immediate extension. I give the proof of the theorem of wider integrability and of the uniformity of this integrability for the set of all subintervals of the interval of integration by a process somewhat different from du Bois-Reymond's process and in a desirably explicit form.

This notion of wider integrability I have found of use in effecting a certain extension of the second mean value theorem of the integral calculus.

2. Du Bois-Reymond's First Integrability Theorem. *If the function*

$$F(y_1, \ldots, y_m)$$

is a continuous function of its m *arguments* $y_1, \ldots, y_m$ *at all points* $Y = (y_1, \ldots, y_m)$ *of the region* R*:*

$$(1) \qquad a_1 \leqq y_1 \leqq \beta_1, \ldots, \quad a_g \leqq y_g \leqq \beta_g, \ldots, \quad a_m \leqq y_m \leqq \beta_m;$$

if, furthermore, the functions $\phi_1(x), \ldots, \phi_m(x)$ *are properly integrable on the x-interval* $a \ldots b$ *and take on (only) values which satisfy the conditions:*

$$(2) \qquad \alpha_1 \leqq \phi_1(x) \leqq \beta_1, \ldots, \alpha_m \leqq \phi_m(x) \leqq \beta_m \qquad (a \leqq x \leqq b);$$

then, the function

$$f(x) = F(\phi_1(x), \ldots, \phi_m(x))$$

is properly integrable on the x-interval $a \ldots b$.

3. Preliminaries of the Extension. The function $F(y_1, \ldots, y_m)$ is continuous on the closed region R, and hence on that region it has definite upper and lower limits (or bounds):

$$\overline{F}_R = \underset{Y|R}{\overline{\mathrm{L}}} F(y_1, \ldots, y_m), \qquad \underline{F}_R = \underset{\overline{Y|R}}{\mathrm{L}} F(y_1, \ldots, y_m),$$

and it is uniformly continuous; that is, for every ϵ a δ_ϵ exists such that

$$|F(y'_1, \ldots, y'_m) - F(y''_1, \ldots, y''_m)| < \epsilon$$

for every two points Y', Y'' of the region R for which

$$|y'_1 - y''_1| < \delta_\epsilon, \ldots, |y'_m - y''_m| < \delta_\epsilon.$$

The function $f(x)$ is, on the interval $a \,.\,.\, b$, a single valued function with definite upper and lower bounds $\overline{f}_{ab}, \underline{f}_{ab}$ which lie on the interval $\underline{F}_R \,.\,.\, \overline{F}_R$.

We consider a partition π of the interval $a \,.\,.\, b$ into a finite number (n) of intervals* $x_0 x_1, \ldots, x_{n-1} x_n (x_0 = a, x_n = b)$ and the corresponding general sum

$$f_\pi = f_1 \delta_1 + \ldots + f_n \delta_n,$$

where $\delta_k = x_k - x_{k-1}$ and f_k is any value lying on the interval $\underline{f}_k \ldots \overline{f}_k$ bounded by the upper and lower limits $\overline{f}_k, \underline{f}_k$ of $f(x)$ on the x-interval $x_{k-1} \ldots x_k$.

Du Bois-Reymond's theorem affirms the existence of the limit:

$$\underset{\delta_k = 0}{\mathrm{L}} f_\pi,$$

in the usual sense of the theory of proper definite integrals.

Now we have as definition

$$f(x) = F(\phi_1(x), \ldots, \phi_m(x)),$$

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* For brevity, here and below in several limit formulas an interval $a' \ldots b'$ is denoted by $a'b'$.

and so

$$\bar{f}_k = \mathop{\bar{\mathrm{L}}}_{x\,|\,x_{k-1}x_k} F(\phi_1(x), \ldots, \phi_m x)), \qquad \underline{f}_k = \mathop{\underline{\mathrm{L}}}_{x\,|\,x_{k-1}x_k} F(\phi_1(x), \ldots, \phi_m(x)).$$

We consider the more inclusive or wider limits

$$\bar{f}^*_k = \bar{\mathrm{L}}\, F(\phi_{1k}, \ldots, \phi_{mk}), \qquad \underline{f}^*_k = \underline{\mathrm{L}}\, F(\phi_{1k}, \ldots, \phi_{mk}),$$

that is, the upper and lower limits of the set of all values of $F(\phi_{1k}, \ldots, \phi_{mk})$ obtained by allowing each argument ϕ_{gk} $(g = 1, \ldots, m)$ independently to run through the interval $\underline{\phi}_{gk} \ldots \bar{\phi}_{gk}$ bounded by the upper and lower limits of $\phi_g(x)$ on the x-interval $x_{k-1} \ldots x_k$; and we denote by f^*_k any number of the interval $\underline{f}^*_k \ldots \bar{f}^*_k$ and consider the wider general sum:

$$f^*_\pi = f^*_1\delta_1 + \ldots + f^*_n\delta_n.$$

Thus every partition π of $a \ldots b$ determines such sums f^*_π connected with $a \ldots b$, and the theorem of the wider integrability affirms the existence of the limit in like sense of these wider general sums and the relation

$$\mathop{\mathrm{L}}_{\delta_k = 0} f^*_\pi = \mathop{\mathrm{L}}_{\delta_k = 0} f_\pi.$$

The partition π of $a \ldots b$ effects a partition π' of a subinterval $a' \ldots b'$ of $a \ldots b$ into (at most) n partial intervals of lengths $\delta'_1, \ldots, \delta'_n$ where then $0 \leqq \delta'_k \leqq \delta_k$, $\delta'_1 + \ldots + \delta'_n = b' - a'$. We consider the general sum

$$f^{b'}_{a'\pi} = f_1\delta'_1 + \cdots + f_n\delta'_n$$

and the wider general sum

$$f^{b'*}_{a'\pi} = f^*_1\delta'_1 + \cdots + f^*_n\delta'_n.$$

From these general sums we obtain the corresponding upper sums by replacing throughout f_k, f^*_k by $\bar{f}_k, \bar{f}^*_k$; and similarly we obtain the corresponding lower sums.

In these notations it is evident that

$$f_\pi = f^b_{a\pi}, \quad f^*_\pi = f^{b*}_{a\pi},$$

and that, for a given partition π and interval $a' \ldots b'$ and corresponding partition π' of $a' \ldots b'$,

$$\underline{f}^{b'}_{a'\pi} \leqq \underline{f}^{b'}_{a'\pi'} \leqq \bar{f}^{b'}_{a'\pi'} \leqq \bar{f}^{b'}_{a'\pi},$$

and so that every sum $\int_{a'\pi'}^{b'}$, is a sum $\int_{a'\pi}^{b'}$, while the converse is in general not true.

We have always the relations:

$$0 \leqq \overline{\int}_{a'\pi'}^{b'} - \underline{\int}_{a'\pi'}^{b'} \leqq \overline{\int}_{a'\pi}^{b'} - \underline{\int}_{a'\pi}^{b'} \leqq \overline{\int}_{a\pi}^{b} - \underline{\int}_{a\pi}^{b},$$

with similar relations for the wider sums.

4. The more intense First Integrability Theorem. *The limits*

$$\underset{\delta_k = 0}{\mathrm{L}} \int_{a'\pi}^{b'*}$$

of the wider sums $\int_{a'\pi}^{b'*}$ *with respect to the various intervals* $a' \ldots b'$ *of* $a \ldots b$ *exist, and the convergence is uniform, for the set of all such intervals* $a' \ldots b'$. *The limiting values are of course*

$$\underset{\delta_k = 0}{\mathrm{L}} \int_{a'\pi'}^{b'} = \int_{a'}^{b} f(x)\,dx \qquad (a \leqq a' < b' \leqq b).$$

Remark. The theorem, in so far as it concerns the sums $\int_{a'\pi}^{b'*}$ instead of the sums $\int_{a'\pi'}^{b'*}$, affirms somewhat more than the uniformity of the wider integrability.

Proof. We shall exhibit for a given ϵ a partition π_ϵ of $a \ldots b$ such that

$$\overline{\int}_{a\pi_\epsilon}^{b*} - \underline{\int}_{a\pi_\epsilon}^{b*} < \epsilon.$$

From this inequality, in view of the final remark of §3, the theorem follows by considerations similar to certain considerations[*] of the usual theory of definite integrals.

The m functions $\phi_g(x)$ $(g = 1, \ldots, m)$ are integrable from a to b. Hence, by Riemann's necessary condition, there exists for each function $\phi_g(x)$ with respect to any two given positive numbers σ, λ a partition $\pi_{g\sigma\lambda}$ of $a \ldots b$ such that the sum of those intervals of $\pi_{g\sigma\lambda}$ on which the oscillation of $\phi_g(x)$ is greater than σ is itself less than λ. By superposition of any m such partitions $\phi_{g\sigma\lambda}$ $(g = 1, \ldots, m)$ we have a partition $\pi_{\sigma\lambda}$ such that on every interval, apart from intervals of total length less than $m\lambda$, the oscillation of each function $\phi_g(x)$ $(g = 1, \ldots, m)$ is at most σ. Taking for σ the $\delta_{\epsilon'}$ connected with the uniform continuity of $F(y_1, \ldots, y_m)$, ϵ' being a positive number subject to later determination, we see that on every interval of $\pi_{\sigma\lambda}$, apart from the intervals specified, the corresponding oscillation $\overline{f}_k - \underline{f}_k$ is less than ϵ' while on every excepted interval the oscillation $\overline{f}_k - \underline{f}_k$ is at most $\overline{F}_R - \underline{F}_R$. Hence for this partition $\pi_{\sigma\lambda}$ $(\sigma = \delta_{\epsilon'})$ we have

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[*] *Cf.* the theorem of Dini, *l. c.*, §185, p. 330.

$$\overline{\int}_{a\,\pi_{\sigma\lambda}}^{b*} - \underline{\int}_{a\,\pi_{\sigma\lambda}}^{b*} < \epsilon'(b-a) + m\lambda\,(\overline{F}_R - \underline{F}_R).$$

Thus, if we set, with respect to the given ϵ,

$$\epsilon' = \frac{1}{2}\frac{\epsilon}{b-a}, \qquad \sigma = \delta_{\epsilon'}, \qquad \lambda = \frac{1}{2m}\frac{\epsilon}{\overline{F}_R - \underline{F}_R + 1},$$

we have in a partition $\pi_{\sigma\lambda}$ the desired partition π_ϵ.

5. An Example in Contravention of du Bois-Reymond's Second Integrability Theorem.

The Theorem. If the function $f(x)$ is properly integrable from a to b and has the limits $\overline{f}_{ab}$, $\underline{f}_{ab}$, and if the function $g(y)$ is properly integrable from $\underline{f}_{ab}$ to $\overline{f}_{ab}$, then the compound function $g(f(x))$ is properly integrable from a to b.

The Example. We take the x-interval $0 \ldots 1$ and consider the exhibition of a number x to the base 3:

$$x = \sum_{r=0}^{\infty} i_{r,x} 3^{-r}$$

where each digit $i_{r,x}$ is 0, 1, or 2; we exclude the repetend 2, and thus have a unique exhibition of every number x $(0 \leqq x \leqq 1)$.

The function $f(x)$ for x with a terminating exhibition:

$$x = \sum_{r=0}^{r_0} i_{r,x} 3^{-r}, \qquad (i_{r_0,x} \neq 0)$$

shall have the value

$$f(x) = (-1)^{i_{r_0,x}}\, 3^{-r_0},$$

while it shall be 0 for all other values of x. This function is properly integrable from 0 to 1, and it has everywhere densely values both positive and negative.

The function $g(y)$:

$$g(y) = \begin{Bmatrix} +1 & y > 0 \\ 0 & y = 0 \\ -1 & y < 0 \end{Bmatrix},$$

is properly integrable on every finite interval.

The function $h(x) = g(f(x))$ for $0 \leqq x \leqq 1$ has the values:

$$h(x) = (-1)^{i_{r_0,x}}, \quad \text{or} \quad 0,$$

according as x has or has not a terminating exhibition. Thus it has values $0, +1, -1$ each everywhere densely. It is accordingly not properly integrable.

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